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EFFICIENT DETECTION OF PERIODIC ORBITS IN CHAOTIC SYSTEMS BY STABILISING TRANSFORMATIONS *

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Abstract. An algorithm for detecting periodic orbits in chaotic systems [Phys. Rev. E, 60 (1999), pp. 6172–6175], which combines the set of stabilising transformations proposed by Schmelcher and Diakonos [Phys. Rev. Lett., 78 (1997), pp. 4733–4736] with a modified semi-implicit Euler iterative scheme and seeding with periodic orbits of neighbouring periods, has been shown to be highly efficient when applied to low-dimensional systems. The difficulty in applying the algorithm to higher-dimensional systems is mainly due to the fact that the number of the stabilising transformations grows extremely fast with increasing system dimension. Here we analyse the properties of stabilising transformations and propose an alternative approach for constructing a smaller set of transformations. The performance of the new approach is illustrated on the four-dimensional kicked double rotor map and the six-dimensional system of three coupled Hénon maps.

Key words. unstable periodic orbits; kicked double rotor map; coupled Hénon maps

AMS subject classifications. 65P99, 65H20, 90C53

1. Introduction.

Periodic orbits play an important role in the analysis of various types of dynamical system. In systems with chaotic behaviour, unstable periodic orbits (UPOs) form a "skeleton" for chaotic trajectories [3]. A well regarded definition of chaos [7] requires the existence of an infinite number of UPOs that are dense in the chaotic set. Different geometric and dynamical properties of chaotic sets, such as natural measure, Lyapunov exponents, fractal dimensions, entropies [19], can be determined from the location and stability properties of the embedded UPOs. Periodic orbits are central to the understanding of quantum-mechanical properties of nonseparable systems: the energy level density of such systems can be expressed in a semiclassical approximation as a sum over the UPOs of the corresponding classical system [9]. Topological description of a chaotic attractor also benefits from the knowledge of periodic orbits. For example, a large set of periodic orbits is highly constraining to the symbolic dynamics and can be used to extract the location of a generating partition [5, 22]. The significance of periodic orbits for the experimental study of dynamical systems has been demonstrated in a wide variety of systems [16], especially for the purpose of controlling chaotic dynamics [20] with possible application in communication [2].

It is therefore not surprising that much effort has been put into the development of methods for locating periodic solutions in different types of dynamical systems. In a limited number of cases, this can be achieved due to the special structure of the systems. Examples include the Biham-Wenzel method applicable to Hénon-like maps [1], or systems with known and well ordered symbolic dynamics [11]. For generic systems, however, most methods described in the literature use some type of an iterative scheme that, given an initial condition (seed), converges to a periodic orbit of the chaotic system. In order to locate all UPOs with a given period p, the convergence basin of each orbit for the chosen iterative scheme must contain at least one seed. The seeds are often chosen either at random from within the region of interest, from a regular grid, or from a chaotic trajectory with or without close recurrences. Typically, the iterative scheme is chosen from one of the "globally" convergent methods of quasi-

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Newton or secant type. However, experience suggests that even the most sophisticated methods of this type suffer from a common problem: with increasing period, the basin size of the UPOs becomes so small that placing a seed within the basin with one of the above listed seeding schemes is practically impossible [18].

A different approach, which appears to effectively deal with the problem of reduced basin sizes has been proposed by Schmelcher and Diakonos (SD) [25, 26]. The basic idea is to transform the dynamical system in such a way that the UPOs of the original system become stable and can be located by simply following the evolution of the transformed dynamical system. That is, to locate period-p orbits of a discrete dynamical system

$$U\colon \quad x_{j+1} = f(x_j), \quad f\colon \mathbb{R}^n \mapsto \mathbb{R}^n \,, \tag{1.1}$$

one considers an associated flow

$$\Sigma\colon \quad \frac{dx}{ds} = Cg(x)\,, \tag{1.2}$$

where $g(x) = f^p(x) - x$ and C is an $n \times n$ constant orthogonal matrix. It is easy to see that map $f^p(x)$ and flow Σ have identical sets of fixed points for any C, while C can be chosen such that unstable period-p orbits of U become stable fixed points of Σ.

Since it is not generally possible to choose a single matrix C that would stabilise all UPOs of U, the idea is to find the smallest possible set of matrices $\mathcal{C} = \{C_k\}_{k=1}^K$, such that, for each UPO of U, there is at least one matrix $C \in \mathcal{C}$ that transforms the unstable orbit of U into a stable fixed point of Σ. To this end, Schmelcher and Diakonos have put forward the following conjecture [25]:

Conjecture 1. *Let $\mathcal{C}_{\mathrm{SD}}$ be the set of all $n \times n$ orthogonal matrices with only ± 1 non-zero entries. Then, for any $n \times n$ non-singular real matrix G, there exists a matrix $C \in \mathcal{C}_{\mathrm{SD}}$ such that all eigenvalues of the product CG have negative real parts.*

Observation 1. *The set $\mathcal{C}_{\mathrm{SD}}$ forms a group isomorphic to the Weyl group B_n [14], i.e. the symmetry group of an n-dimensional hypercube. The number of matrices in $\mathcal{C}_{\mathrm{SD}}$ is $K = 2^n n!$.*

The above conjecture has been verified for $n \leq 2$ and appears to be true for $n > 2$, but, thus far, no proof has been presented. According to this conjecture, any periodic orbit, whose stability matrix does not have eigenvalues equal to one, can be transformed into a stable fixed point of Σ with $C \in \mathcal{C}_{\mathrm{SD}}$. In practice, to locate periodic orbits of the map U, we try to integrate the flow Σ from a given initial condition (seed) using different matrices from the set $\mathcal{C}_{\mathrm{SD}}$. Some of the resulting trajectories will converge to fixed points, while others will fail to do so, either leaving the region of interest or failing to converge within a specified number of steps.

The main advantage of the SD approach is that the convergence basins of the stabilised UPOs appear to be much larger than the basins produced by other iterative schemes [26, 15, 6], making it much easier to select a useful seed. Moreover, depending on the choice of the stabilising transformation, the SD method may converge to several different UPOs from the same seed.

The flow Σ can be integrated by any off-the-shelf numerical integrator. Schmelcher and Diakonos have enjoyed considerable success using a simple Euler method. However, the choice of the integrator for this problem is governed by considerations very different from those typically used to construct an ODE solver. Indeed, to locate

a fixed point of the flow, it may not be very efficient to follow the flow with some prescribed accuracy. Therefore, local error considerations, for example, are not as important. Instead, the goal is to have a solver that can reach the fixed point in as few integration steps as possible. In fact, as shown by Davidchack and Lai [4], the efficiency of the method can be improved dramatically when the solver is constructed specifically with the above goal in mind. In particular, recognizing the typical stiffness of the flow Σ, Davidchack and Lai have proposed a modified semi-implicit Euler method:

$$x_{j+1} = x_j + [\beta s_j C^{\mathsf{T}} - G_j]^{-1} g(x_j) \ , \tag{1.3}$$

where $\beta > 0$ is a scalar parameter, $s_j \equiv ||g(x_j)||$ is an L_2 norm, $G_j \equiv Dg(x_j)$ is the Jacobian matrix, and "T" denotes transpose. Note that, away from the root of g, the above iterative scheme is a semi-implicit Euler method with step size $h = (\beta s_j)^{-1}$ and, therefore, can follow the flow Σ with a much larger step size than an explicit integrator (e.g. Euler or Runge-Kutta). Close to the root, the proposed scheme can be shown to converge quadratically [15], analogous to the Newton-Raphson method.

Another important ingredient of the algorithm presented in [4] is the seeding with already detected periodic orbits of neighbouring periods. This seeding scheme appears to be superior to the typically employed schemes and enables fast detection of plausibly all[1] periodic orbits of increasingly larger periods in generic low-dimensional chaotic systems. For example, for the Ikeda map at traditional parameter values, the algorithm presented in [4] was able to locate plausibly all periodic orbits up to period 22 for a total of over 10^6 orbit points. Obtaining a comparable result with generally employed techniques requires an estimated 10^5 larger computational effort.

While the stabilisation approach is straightforward for relatively low-dimensional systems, direct application to higher-dimensional systems is much less efficient due to the rapid growth of the number of matrices in $\mathcal{C}_{\mathrm{SD}}$. Even though it appears that, in practice, far fewer transformations are required to stabilise all periodic orbits of a given chaotic system [21], the sufficient subset of transformations is not known a priori. It is also clear that the route of constructing a universal set of transformations is unlikely to yield substantial reduction in the number of such transformations. For instance, a smaller set of universal transformations with $K = (n+1)!$, which is isomorphic to the Weyl group A_n, is sufficient to stabilise all types of periodic orbits for $n < 4$, but can be shown to fail for certain types of orbits when $n \geq 4$. Therefore, a more promising way of using stabilising transformations for locating periodic orbits in high-dimensional systems is to design such transformations based on the information about the properties of the system under investigation.

In this Article, we propose to construct stabilising transformations based on the knowledge of the stability matrices of already detected periodic orbits (used as seeds). The advantage of our approach is in a substantial reduction of the number of transformations, which increases the efficiency of the detection algorithm, especially in the

[1] It is not possible to prove, within our approach, the completeness of the detected sets of UPOs. Rather, our assertion of completeness is based on the plausibility argument. The following three criteria are used for the validation of the argument:
i) Methods based on rigorous numerics (e.g. in [8]) have located the same UPOs in cases where such comparison is possible (usually for low periods, since these methods are less efficient).
ii) Our search strategy scales with the period p (see §4 and [6]). If we can tune it to locate all UPOs for low periods (where we can verify the completeness using (i)), it is likely (but not provably) capable of locating all UPOs of higher periods as well.
iii) For maps with symmetries, we test the completeness by verifying that all the symmetric partners for all detected UPOs have been found (see §4.1 and §4.2).

case of higher-dimensional systems. The layout of the paper is as follows. In §2 we study the properties of the stabilising transformations for $n = 2$ and their relationship to the properties of the stability matrix of a periodic orbit. In §3 we extend the analysis to higher-dimensional systems and show how to construct stabilising transformations using the knowledge of the stability matrices of already detected periodic orbit points. In particular, we argue that the stabilising transformations depend essentially on the signs of unstable eigenvalues and the directions of the corresponding eigenvectors of the stability matrices. Section 4 illustrates the application of the new stabilising transformations to the detection of periodic orbits in a four-dimensional kicked double rotor map and a six-dimensional system of three coupled Hénon maps. We conclude with the summary and discussion of possible further developments of the stabilising transformations approach in §5.

2. Stabilising transformations in two dimensions. The stability of a fixed point x^* of the flow Σ is determined by the real parts of the eigenvalues of the matrix CG, where $G \equiv Dg(x^*)$ is the Jacobian matrix of $g(x)$ evaluated at x^*. For x^* to be a stable fixed point of Σ, the matrix C has to be such that all the eigenvalues of CG have negative real parts. In order to understand what properties of G determine the choice of a particular stabilising transformation C, we use the following parametrisation for the general two-dimensional orthogonal matrices:

$$C_{s,\alpha} = \begin{pmatrix} s\cos\alpha & \sin\alpha \\ -s\sin\alpha & \cos\alpha \end{pmatrix} \tag{2.1}$$

where $s = \pm 1$ and $-\pi < \alpha \le \pi$. When $\alpha = -\pi/2,\ 0,\ \pi/2$, or π, we obtain the set of matrices $\mathcal{C}_{\mathrm{SD}}$. For example, $C_{1,\pi/2} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $C_{-1,\pi} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

If we write $G \equiv g_{ij}$, $(i, j = 1, 2)$, then the eigenvalues of $C_{s,\alpha}G$ are given by the following equations:

$$\sigma_{1,2} = -A\cos(\alpha - \theta) \pm \sqrt{A^2\cos^2(\alpha - \theta) - s\det G} \tag{2.2}$$

where $\det G = g_{11}g_{22} - g_{12}g_{21}$, $A = \frac{1}{2}\sqrt{(sg_{11} + g_{22})^2 + (sg_{12} - g_{21})^2}$, and

$$\tan\theta = \frac{sg_{12} - g_{21}}{-sg_{11} - g_{22}}\,, \qquad -\pi < \theta \le \pi\,. \tag{2.3}$$

Note that the signs of the numerator and denominator are significant for defining angle θ in the specified range and should not be canceled out. It is clear from Eq.(2.2) that both eigenvalues have negative real parts when

$$s = \bar{s} \equiv \operatorname{sgn}\det G, \qquad \text{and} \qquad |\alpha - \theta| < \tfrac{\pi}{2}\,, \tag{2.4}$$

provided that $\det G \ne 0$. This result proves the validity of Conjecture 1 for $n = 2$. Moreover, it shows that there are typically two matrices in $\mathcal{C}_{\mathrm{SD}}$ that stabilise a given fixed point.

Parameter θ clearly plays an important role in the above analysis. The following theorems show its relationship to the eigenvalues and eigenvectors of the stability matrix of a fixed point.

THEOREM 2.1. *Let x^* be a saddle fixed point of $f^p(x) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ whose stability matrix $Df^p(x^*)$ has eigenvalues $\lambda_{1,2}$ such that $|\lambda_2| < 1 < |\lambda_1|$ and eigenvectors defined by the polar angles $0 \le \phi_{1,2} < \pi$, i.e. $v_{1,2} = (\cos\phi_{1,2}, \sin\phi_{1,2})^{\mathsf{T}}$. Then the following*

is true for angle θ defined in Eq. (2.3):
<u>*Case 1:* $\lambda_1 < -1$</u>

$$\theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) . \tag{2.5}$$

Moreover, if $|\lambda_1| \gg 1$*, then*

$$\theta \approx (\phi_1 - \phi_2)(\text{mod } \pi) - \tfrac{\pi}{2} . \tag{2.6}$$

<u>*Case 2:* $\lambda_1 > 1$</u>

$$\theta = \begin{cases} \frac{3\pi}{2} - \phi_1 - \phi_2 , & 0 < \phi_1 - \phi_2 < \pi , \\ \frac{\pi}{2} - \phi_1 - \phi_2 , & -\pi < \phi_1 - \phi_2 < 0 . \end{cases} \tag{2.7}$$

Proof. Matrix $G = Df^p(x^*) - I$, where I is the identity matrix, can be written as follows:

$$G \equiv \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \cos\phi_1 & \cos\phi_2 \\ \sin\phi_1 & \sin\phi_2 \end{pmatrix} \begin{pmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 - 1 \end{pmatrix} \begin{pmatrix} \cos\phi_1 & \cos\phi_2 \\ \sin\phi_1 & \sin\phi_2 \end{pmatrix}^{-1} \tag{2.8}$$

<u>Case 1</u>: Since $\det G = (\lambda_1 - 1)(\lambda_2 - 1) > 0$ we set $s = 1$ and obtain from Eq. (2.3):

$$\tan\theta = \frac{(\lambda_1 - \lambda_2)\cot(\phi_1 - \phi_2)}{2 - \lambda_1 - \lambda_2} , \tag{2.9}$$

where, just like in Eq. (2.3), as well as in Eqs. (2.10) and (2.13) below, the signs of the numerator and denominator should not be canceled out. Since $2 - \lambda_1 - \lambda_2 > 0$, we have that $\cos\theta > 0$ or

$$\theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) .$$

For $|\lambda_1| \gg 1$, Eq. (2.9) yields:

$$\tan\theta \approx -\cot(\phi_1 - \phi_2) ,$$

and, given Eq. (2.5), the result in Eq. (2.6) immediately follows.
<u>Case 2</u>: In this case $\det G = (\lambda_1 - 1)(\lambda_2 - 1) < 0$, so, from Eq. (2.3) with $s = -1$:

$$\begin{aligned} \tan\theta &= \frac{(\lambda_2 - \lambda_1)\cos(\phi_1 + \phi_2)/\sin(\phi_1 - \phi_2)}{(\lambda_2 - \lambda_1)\sin(\phi_1 + \phi_2)/\sin(\phi_1 - \phi_2)} \\ &= \frac{-\cos(\phi_1 + \phi_2)/\sin(\phi_1 - \phi_2)}{-\sin(\phi_1 + \phi_2)/\sin(\phi_1 - \phi_2)} , \end{aligned} \tag{2.10}$$

since $\lambda_2 - \lambda_1 < 0$. The result in Eq. (2.7) follows. ∎

THEOREM 2.2. *Let x^* be a spiral fixed point of $f^p(x) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ whose stability matrix $Df^p(x^*)$ has eigenvalues $\lambda_{1,2} = \lambda \pm \mathrm{i}\omega$. Then*

$$\begin{aligned} &\theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) && \text{if} \quad \lambda < 1 , \\ &\theta \in \left(-\pi, -\tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right) && \text{if} \quad \lambda > 1 . \end{aligned} \tag{2.11}$$

Proof. The stability matrix can be decomposed as follows:

$$Df^p(x^*) = \begin{pmatrix} \cos\phi & e^\eta \\ \sin\phi & 0 \end{pmatrix} \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix} \begin{pmatrix} \cos\phi & e^\eta \\ \sin\phi & 0 \end{pmatrix}^{-1}, \tag{2.12}$$

where $\eta \in \mathbb{R}$. Given that $G = Df^p(x^*) - I$, we have from Eq. (2.3):

$$\tan\theta = \frac{-\omega\cosh\eta/\sin\phi}{1-\lambda}. \tag{2.13}$$

The result in Eq. (2.11) follows from the sign of the denominator. ∎

The key message of the above theorems is that the stabilising transformation matrix depends mostly on the directions of the eigenvectors and the signs of the unstable[2] eigenvalues (or their real parts), and only marginally on the actual magnitudes of the eigenvalues. This means that a transformation that stabilises a given fixed point x^* of f^p will also stabilise fixed points of all periods with similar directions of eigenvectors and signs of the unstable eigenvalues. In the next Section, we will show how this observation can be used to construct stabilising transformations for efficient detection of periodic orbits in systems with $n > 2$.

3. Extension to higher-dimensional systems. To extend the analysis of the preceding Section to higher-dimensional systems, we note that the matrix $C_{\bar{s},\theta}$, as defined by Eqs. (2.1), (2.3), and (2.4), is closely related to the orthogonal part of the *polar decomposition* of G [10]. Recall that any non-singular $n \times n$ matrix can be uniquely represented as a product

$$G = QB, \tag{3.1}$$

where Q is an orthogonal matrix and B is a symmetric positive definite matrix. The following theorem provides the link between $C_{\bar{s},\theta}$ and Q for $n = 2$:

THEOREM 3.1. *Let $G \in \mathbb{R}^{2\times 2}$ be a non-singular matrix with the polar decomposition $G = QB$, where Q is an orthogonal matrix and B is a symmetric positive definite matrix. Then matrix $C_{\bar{s},\theta}$, as defined by Eqs. (2.1), (2.3) and (2.4), is related to Q as follows:*

$$C_{\bar{s},\theta} = -Q^{\mathsf{T}} \tag{3.2}$$

Proof. Since $C_{\bar{s},\theta}$ is an orthogonal matrix by definition, it is sufficient to prove that $C_{\bar{s},\theta}G$ is symmetric negative definite. Then, by the uniqueness of the polar decomposition, it must be equal to $-B$.

Denote by b_{ij} the element in the i-th row and j-th column of $C_{\bar{s},\theta}G$. We must show that $b_{12} = b_{21}$. Using Eq. (2.3), we have that

$$\begin{aligned} b_{12} &= \bar{s}g_{12}\cos\theta + g_{22}\sin\theta \\ &= \left[\bar{s}g_{12} + g_{22}\frac{\bar{s}g_{12} - g_{21}}{-\bar{s}g_{11} - g_{22}}\right]\cos\theta \\ &= \left[\frac{g_{11}g_{12} + g_{21}g_{22}}{\bar{s}g_{11} + g_{22}}\right]\cos\theta, \end{aligned} \tag{3.3}$$

[2] That is, eigenvalues whose magnitude is larger than one.

and similarly

$$b_{21} = g_{21}\cos\theta - \bar{s}g_{11}\sin\theta = \left[g_{21} - \bar{s}g_{11}\frac{\bar{s}g_{12} - g_{21}}{-\bar{s}g_{11} - g_{22}}\right]\cos\theta = \left[\frac{g_{11}g_{12} + g_{21}g_{22}}{\bar{s}g_{11} + g_{22}}\right]\cos\theta\,, \tag{3.4}$$

hence the matrix $C_{\bar{s},\theta}G$ is symmetric. Since, by definition, θ and $\bar{s}$ are chosen such that the eigenvalues of $C_{\bar{s},\theta}G$ are negative, the matrix $C_{\bar{s},\theta}G$ is negative definite. Finally, by the uniqueness of the polar decomposition,

$$C_{\bar{s},\theta}G = -B = -Q^{\mathsf{T}}G\,,$$

which completes the proof. □

For $n > 2$, we can always use the polar decomposition to construct a transformation that will stabilise a given fixed point. Indeed, if a fixed point x^* of an n-dimensional flow has a non-singular matrix $G \equiv Dg(x^*)$, then we can calculate the polar decomposition $G = QB$ and use

$$C = -Q^{\mathsf{T}}\,, \tag{3.5}$$

to stabilise x^*. Moreover, by analogy with the two-dimensional case, we can expect that the same matrix C will also stabilise fixed points $\tilde{x}$ with the matrix $\tilde{G} \equiv Dg(\tilde{x})$, as long as the orthogonal part $\tilde{Q}$ of the polar decomposition $\tilde{G} = \tilde{Q}\tilde{B}$ is sufficiently close to Q. More precisely,

Observation 2. *C will stabilise $\tilde{x}$, if all eigenvalues of the product $Q\tilde{Q}^{\mathsf{T}}$ have positive real parts.*

We base this observation on the following corollary of Lyapunov's stability theorem [13]:

Corollary 3.2. *Let $B \in \mathbb{R}^{n\times n}$ be a positive definite symmetric matrix. If $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix such that all its eigenvalues have positive real parts, then all the eigenvalues of the product QB have positive real parts as well.*

Proof. According to Lyapunov's theorem, a matrix $A \in \mathbb{R}^{n\times n}$ has all eigenvalues with positive real parts if and only if there exists a symmetric positive definite $G \in \mathbb{R}^{n\times n}$ such that $GA + A^{\mathsf{T}}G = H$ is positive definite.

Let $A = QB$ and let's choose G in the form $G = \frac{1}{2}QB^{-1}Q^{\mathsf{T}}$. Since B is positive definite, its inverse B^{-1} is also positive definite, and, since G and B^{-1} are related by a congruence transformation, according to Sylvester's inertia law [12], G is also positive definite. Now,

$$GQB + (QB)^{\mathsf{T}}G = \tfrac{1}{2}QB^{-1}Q^{\mathsf{T}}QB + \tfrac{1}{2}BQ^{\mathsf{T}}QB^{-1}Q^{\mathsf{T}} = \tfrac{1}{2}[Q + Q^{\mathsf{T}}]\,.$$

Therefore, QB has eigenvalues with positive real parts if and only if $\frac{1}{2}[Q + Q^{\mathsf{T}}]$ is positive definite. The proof is completed by observing that, for orthogonal matrices, the eigenvalues of $\frac{1}{2}[Q + Q^{\mathsf{T}}]$ are equal to the real parts of the eigenvalues of Q. □

Note that Observation 2 is a direct generalisation of conditions in Eq. (2.4) which are equivalent to requiring that the eigenvalues of $C_{s,\alpha}C^{\mathsf{T}}_{\bar{s},\theta}$ have positive real parts.

In the scheme where already detected periodic orbits are used as seeds to detect other orbits [4], we can use C in Eq. (3.5) as a stabilising matrix for the seed x^*. Based on the analysis in §2, this will allow us to locate a periodic orbit in the neighbourhood

of x^* with similar invariant directions and the same signs of the unstable eigenvalues. Note, however, that the neighbourhood of the seed x^* can also contain periodic orbits with the similar invariant directions but with some eigenvalues having the opposite sign (i.e. orbits with and without reflections). To construct transformations that would stabilise such periodic orbits, we can determine the eigenvalues and eigenvectors of the stability matrix of x^*

$$Df^p(x^*) = V\Lambda V^{-1}\,, \tag{3.6}$$

where $\Lambda \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues of $Df^p(x^*)$ and V is the matrix of eigenvectors, and then calculate the polar decomposition of the matrix

$$\hat{G} = V(S\Lambda - I)V^{-1}\,, \tag{3.7}$$

where $S = \mathrm{diag}(\pm 1, \pm 1, \ldots, \pm 1)$. Note that, as follows from the analysis in §2 for $n = 2$ and numerical evidence for $n > 2$, changing the sign of a stable eigenvalue will not result in a substantially different stabilising transformation. Therefore, we restrict our attention to the following subset of S:

$$S_{ii} = \begin{cases} \pm 1, & |\lambda_i| > 1\,, \\ \;\;\,1, & |\lambda_i| < 1\,, \end{cases} \qquad \text{for} \quad i = 1, \ldots, n\,. \tag{3.8}$$

For a seed with k real unstable eigenvalues, this results in 2^k possible transformations. Note that, on the one hand, this set is much smaller than $\mathcal{C}_{\mathrm{SD}}$, while, on the other hand, it allows us to target all possible types of periodic orbits that have invariant directions similar to those at the seed.

4. Numerical results. In this Section we illustrate the performance of the new stabilising transformations on a four-dimensional kicked double rotor map [24] and a six-dimensional system of three coupled Hénon maps [23]. Both systems are highly chaotic and the number of UPOs is expected to grow rapidly with increasing period. The goal is to locate *all* UPOs of increasingly larger period. Of course, the completeness of the set of orbits for each period cannot be guaranteed, but it can be established with high degree of certainty by using the plausibility criteria outlined in the Introduction.

In order to start the detection process, we need to have a small set of periodic orbits (of period $p > 1$) that can be used as seeds. Such orbits can be located using, for example, random seeds and the standard Newton-Raphson method (or the scheme in Eq. (1.3) with $\beta = 0$). We can then use these periodic orbits as seeds to construct the stabilising transformations and detect more UPOs with higher efficiency. The process can be iterated until we find no more orbits of a given period. In our previous work [4, 6] we showed that for two-dimensional maps such as Hénon and Ikeda it is sufficient to use period-$(p-1)$ orbits as seeds to locate plausibly all period-p orbits. For higher-dimensional systems, such as those considered in the present work, these seeds may not be sufficient. However, it is always possible to use more seeds by, for example, locating some of the period-$(p+1)$ orbits, which can then be used as seeds to complete the detection of period-p orbits. The following recipe can be used as a general guideline for developing a specific detection scheme for a given system:

1. *Find a set of orbit points of low period using random seeds and the iterative scheme in Eq. (1.3) with $\beta = 0$ (i.e. the Newton-Raphson scheme).*

2. *To locate period-p orbits, first use period-$(p-1)$ orbits as seeds. For each seed x_0, construct 2^k stabilising transformations C using Eqs. (3.6-3.8), where k is the number of unstable eigenvalues of $Df^{p-1}(x_0)$.*

3. *Starting from* x_0 *and with a fixed value of* $\beta > 0$ *use the iteration scheme in Eq. (1.3) to construct a sequence* $\{x_i\}$ *for each of the* 2^k *stabilising transformations. If a sequence converges to a point, check whether it is a new period-p orbit point, and if so, proceed to find a complete orbit by iterating the map* f.

4. *Repeat steps* $2-3$ *for several* β *in order to determine the optimal value of this parameter (see explanation below).*

5. *Repeat steps* $2-4$ *using newly found period-p points as seeds to search for period-*$(p+1)$ *orbits.*

6. *Repeat steps* $2-4$ *using incomplete set of period-*$(p+1)$ *orbits as seeds to find any missing period-p orbits.*

Although we know that the action of β is to increase the basin size of the stabilised points, it is not known *a priori* what values of β to use for a given system and period. Monitoring the fraction of seeds that converge to periodic orbits, we observe that it grows with increasing β until it reaches saturation, indicating that the iterative scheme faithfully follows the flow Σ. On the other hand, larger β translates into smaller integration steps and, therefore, longer iteration sequences. Thus the optimal value of β is just before the saturation point. As demonstrated in our previous work [4] and observed in the numerical examples presented in the following sections, this value appears to scale exponentially with the period and can be estimated based on the information about the detection pattern at lower periods.

The stopping criteria in step 3, which we use in the numerical examples discussed below, are as follows. The search for UPOs is conducted within a rectangular region containing a chaotic invariant set. The sequence $\{x_i\}$ is terminated if (i) x_i leaves the region, (ii) i becomes larger than a pre-defined maximum number of iterations (we use $i > 100 + 5\beta$), (iii) the sequence converges, such that $\|g(x_i)\| < Tol_g$. In cases (i) and (ii) a new sequence is generated from a different seed and/or with a different stabilising matrix. In case (iii) five Newton iterations are applied to x_i to allow convergence to a fixed point to within the round-off error. A point x^* for which $\|g(x^*)\|$ is the smallest is identified with a fixed point of f^p. The maximum round-off error over the set $\mathcal{X}_p$ of all detected period-p orbit points

$$\epsilon_{\max}(p) = \max\{\|g(x^*)\| : x^* \in \mathcal{X}_p\} \tag{4.1}$$

is monitored in order to assess the accuracy of the detected orbits.

To check if the newly detected orbit is different from those already detected, its distance to other orbit points is calculated: if $\|x^* - y^*\|_\infty > Tol_x$ for all previously detected orbit points y^*, then x^* is a new orbit point. Even for a large number of already detected UPOs, this check can be done very quickly by pre-sorting the detected orbit points along one of the system coordinates and performing a binary search for the points within Tol_x of x^*. The infinity norm in the above expression is used for the computational efficiency of this check.

The minimum distance between orbit points

$$d_{\min}(p) = \min\{\|x^* - y^*\|_\infty : x^*, y^* \in \mathcal{X}_p\} \tag{4.2}$$

is monitored and the algorithm is capable of locating all isolated UPOs of a given period p as long as $\epsilon_{\max}(p) < Tol_g \lesssim Tol_x < d_{\min}(p)$. Since typically $\epsilon_{\max}(p)$ increases and $d_{\min}(p)$ decreases with p (see Tables 4.1 and 4.2), the above conditions can be satisfied up to some period, after which higher-precision arithmetics needs to be used in the evaluation of the map. For the numerical examples presented in the following sections we use double-precision computation with $Tol_g = 10^{-6}$ and $Tol_x = 10^{-5}$.

TABLE 4.1

Number $n(p)$ of prime period-p UPOs, and the number $N(p)$ of fixed points of p-times iterated map for the kicked double rotor map. The asterisk for $p = 8$ indicates that this set of orbits is not complete. Parameters $\epsilon_{\max}(p)$ and $d_{\min}(p)$ are defined in Eqs. (4.1) and (4.2).

p	$n(p)$	$N(p)$	$\epsilon_{\max}(p)$	$d_{\min}(p)$
1	12	12	$1.0 \cdot 10^{-14}$	$1.3 \cdot 10^{0}$
2	45	102	$5.9 \cdot 10^{-14}$	$3.4 \cdot 10^{-1}$
3	152	468	$5.8 \cdot 10^{-13}$	$6.2 \cdot 10^{-2}$
4	522	2190	$2.7 \cdot 10^{-12}$	$6.9 \cdot 10^{-3}$
5	2200	11 012	$2.6 \cdot 10^{-11}$	$1.1 \cdot 10^{-3}$
6	9824	59 502	$1.6 \cdot 10^{-10}$	$1.8 \cdot 10^{-4}$
7	46 900	328 312	$9.7 \cdot 10^{-10}$	$9.1 \cdot 10^{-5}$
8*	229 082	1 834 566	$1.2 \cdot 10^{-8}$	$5.5 \cdot 10^{-5}$

4.1. Kicked double rotor map. The kicked double rotor map describes the dynamics of a mechanical system known as the double rotor under the action of a periodic kick [24]. It is a four-dimensional map defined by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} My_n + x_n \pmod{2\pi} \\ Ly_n + c \sin x_{n+1} \end{pmatrix}, \tag{4.3}$$

where $x_n \in \mathbb{S}^2$ are the angle coordinates and $y_n \in \mathbb{R}^2$ are the angular velocities after each kick. Parameters L and M are constant 2×2 matrices that depend on the masses, lengths of rotor arms, and friction at the pivots, while $c \in \mathbb{R}^2$ is a constant vector whose magnitude is proportional to the kicking strength f_0. In our numerical tests we have used the same parameters as in [24], with the kicking strength $f_0 = 8.0$.

The following example illustrates the stabilising properties of the transformations constructed on the basis of periodic orbits. Let us take a typical period-3 orbit point $x^* = (0.6767947, 5.8315697)$, $y^* = (0.9723920, -7.9998313)$ as a seed for locating period-4 orbits. The Jacobian matrix $Df^3(x^*, y^*)$ of the seed has eigenvalues $\Lambda = \mathrm{diag}(206.48, -13.102, -0.000373, 0.000122)$. Therefore, based on the scheme discussed in §3 Eqs. (3.6-3.8), we can construct four stabilising transformations C corresponding to (S_{11}, S_{22}) in Eq. (3.8) being equal to $(+,+)$, $(-,+)$, $(+,-)$ and $(-,-)$. Of the total of 2190 orbit points of period-4 (see Table 4.1), the transformations C_1, C_2, C_3, and C_4 stabilise $\#(1) = 532$, $\#(2) = 544$, $\#(3) = 474$, and $\#(4) = 516$ orbit points, respectively, and these sets of orbits are almost completely non-overlapping. That is, the number of orbits stabilised by both C_1 and C_2 is $\#(1 \cap 2) = 2$. Similarly, $\#(1 \cap 3) = 16$, $\#(1 \cap 4) = 0$, $\#(2 \cap 3) = 0$, $\#(2 \cap 4) = 14$, and $\#(3 \cap 4) = 0$. On the other hand, the number of period-4 orbits stabilised by at least one of the four transformations is $\#(1 \cup 2 \cup 3 \cup 4) = 2034$. This is a typical picture for other seeds of period-3 as well as other periods.

This example provides evidence for the validity of our approach to constructing the stabilising transformations in high-dimensional systems based on periodic orbits. It also shows that, in the case of the double rotor map, a single seed is sufficient for constructing transformations that stabilise majority of the UPOs. Of course, in order to locate the UPOs, we need to ensure that the seeds are in the convergence basins of the stabilised periodic orbits. That is why we need to use more seeds to locate plausibly all periodic orbits of a given period. Still, because of the enlarged basins of the stabilised orbits, the number of seeds is much smaller than that required with iterative schemes that do not use the stabilising transformations.

Compared to the total of 384 matrices in $\mathcal{C}_{\mathrm{SD}}$, we use only two or four transformations for each seed, depending on the number of unstable directions of the seed

orbit points. Yet, the application of the detection scheme outlined in §4 allows us to locate plausibly all periodic orbits of the double rotor map up to period 7. Table 4.1 also includes the number of detected period-8 orbits that were used as seeds to complete the detection of period 7.

The confidence with which we claim to have plausibly complete sets of periodic orbits for each period is enhanced by the symmetry consideration. That is, since the double rotor map is invariant under the change of variables $(x, y) \mapsto (2\pi - x, -y)$, a necessary condition for the completeness of the set of orbits for each period is that for any orbit point (x^*, y^*) the set also contains an orbit point $(2\pi - x^*, -y^*)$. Even though this condition was not used in the detection scheme, we find that the detected sets of orbits (apart from period 8) satisfy this symmetry condition. Of course, this condition is not sufficient to prove the completeness of the detected sets of UPOs, but, combined with the exhaustive search procedure presented above, provides a strong indication of the completeness.

4.2. Coupled Hénon maps. Another system we use to test the efficacy of our approach is a six-dimensional system of three coupled Hénon maps (CHM),

$$x_{n+1}^j = a - (\tilde{x}_n^j)^2 + b x_{n-1}^j, \quad \text{for} \quad j = 1, 2, 3\,, \tag{4.4}$$

where $a = 1.4$ and $b = 0.3$ are the standard parameter values of the Hénon map and the coupling is given by

$$\tilde{x}_n^j = (1 - \epsilon) x_n^j + \frac{1}{2}\epsilon(x_n^{j+1} + x_n^{j-1}), \tag{4.5}$$

with $x_n^0 \equiv x_n^3$ and $x_n^4 \equiv x_n^1$. We have chosen the coupling parameter $\epsilon = 0.15$. Our choice of this system is motivated by the work of Politi and Torcini [23] in which they locate periodic orbits in CHM for a small coupling parameter by extending the method of Biham and Wenzel [1]. This makes the CHM an excellent test system, since we can compare our results against those for the Biham-Wenzel (BW) method. The BW method defines the following artificial dynamics

$$\dot{x}_n^j(t) = (-1)^{s(n,j)}\{x_{n+1}^j(t) - a + [\tilde{x}_n^j(t)]^2 - b x_{n-1}^j(t)\}, \tag{4.6}$$

with $s(n, j) \in \{0, 1\}$. Given the boundary condition $x_{p+1}^j = x_1^j$, the equilibrium states of Eq. (4.6) are the period-p orbits for the CHM. The BW method is based on the property that every equilibrium state of Eq. (4.6) can be made stable by one of the 2^{3p} possible sequences of $s(n, j)$ and, therefore, can be located by simply integrating Eq. (4.6) to convergence starting from the same initial condition $x_n^j = 0.0$. It is also found that, for the vast majority of orbits, each orbit is stabilised by a unique sequence of $s(n, j)$.

In order to reduce the computational effort Politi and Torcini suggest reducing the search to only those sequences $s(n, j)$ which are allowed in the uncoupled system, i.e. with $\epsilon = 0$. This reduction is possible because the introduction of coupling has the effect of pruning some of the orbits found in the uncoupled Hénon map without creating any new orbits.

We have implemented the BW method with both the full search and the reduced search (BW-r) up to as high a period as is computationally feasible (see Table 4.2). In the case of the full search we detect UPOs up to period 8 and in the case of the reduced search up to period 12. The seed $x_n^j = 0.0$ was used for all periods except for period 4, where it was found that with this seed both BW and BW-r located only

Table 4.2

The number of prime UPOs for the system of three coupled Hénon maps (CHM) detected by three different methods: BW – full Biham-Wenzel, BW-r – reduced Biham-Wenzel, ST – our method based on stabilising transformations, Max – maximum number of detected UPOs obtained from all three methods and the system symmetry. See text for details.

p	BW	BW-r	ST	Max	$\epsilon_{\max}(p)$	$d_{\min}(p)$
1	8	8	8	8	$1.3 \cdot 10^{-14}$	$9.9 \cdot 10^{-1}$
2	28	28	28	28	$4.6 \cdot 10^{-14}$	$5.2 \cdot 10^{-1}$
3	0	0	0	0	-	-
4	34	34	40	40	$2.7 \cdot 10^{-8}$	$4.2 \cdot 10^{-2}$
5	0	0	0	0	-	-
6	74	74	72	74	$9.5 \cdot 10^{-10}$	$8.6 \cdot 10^{-3}$
7	28	28	28	28	$1.0 \cdot 10^{-8}$	$5.6 \cdot 10^{-3}$
8	271	271	285	286	$1.1 \cdot 10^{-6}$	$5.5 \cdot 10^{-3}$
9	-	63	64	66	$9.9 \cdot 10^{-7}$	$2.6 \cdot 10^{-4}$
10	-	565	563	568	$1.3 \cdot 10^{-8}$	$4.1 \cdot 10^{-4}$
11	-	272	277	278	$7.1 \cdot 10^{-9}$	$5.4 \cdot 10^{-4}$
12	-	1972	1999	1999	$2.5 \cdot 10^{-6}$	$4.3 \cdot 10^{-4}$
13*	-	-	1079	-	$8.6 \cdot 10^{-8}$	$4.0 \cdot 10^{-4}$
14*	-	-	6599	-	$2.3 \cdot 10^{-6}$	$3.5 \cdot 10^{-4}$
15*	-	-	5899	-	$7.0 \cdot 10^{-6}$	$1.5 \cdot 10^{-4}$

28 orbits. We found a maximum of 34 orbits using the seed $x_n^j = 0.5$. It is possible that more orbits can be found with different seeds for other periods as well, but we have not investigated this. The example of period 4 illustrates that, unlike for a single Hénon map, the Biham-Wenzel method fails to detect all orbits from a single seed.

Even though our approach (labeled “ST” in Table 4.2) is general and does not rely on the special structure of the Hénon map, its efficiency far surpasses the full BW method and is comparable to the reduced BW method. Except for periods 6 and 10, the ST method locates the same or larger number of orbits.[3]

Unlike the double rotor map, the CHM possesses very few periodic orbits for small p, particularly for odd values of p. Therefore, we found that the direct application of the detection strategy outlined at the beginning of §4 would not allow us to complete the detection of even period orbits. Therefore, for even periods p we also used $p+2$ as seeds and, in case of period 12, a few remaining orbits were located with seeds of period 15. We did not attempt to locate a maximum possible number of UPOs for $p > 12$. The numbers of such orbits (labeled with asterisks) are listed in Table 4.2 for completeness.

As with the double rotor map, we used the symmetry of the CHM to test the completeness of the detected sets of orbits. It is clear from the definition of the CHM that all its UPOs are related by the permutation symmetry (i.e., six permutations of indices j). The column labeled “Max” in Table 4.2 lists the maximum number of UPOs that we were able to find using all three methods and applying the permutation symmetry to find any UPOs that might have been missed. As can be seen in Table 4.2, only a few orbits remained undetected by the ST method.

Concluding this Section, we would like to point out that the high efficiency of the proposed method is primarily due to the fact that each stabilising transformation constructed based on the stability properties of the seed orbit substantially increases the basins of convergence of orbits stabilised by this transformation. This is apparent

[3]The precise reason for the failure of the ST method to detect all period 6 and 10 orbits needs further investigation. We believe that the orbits that were not detected have uncharacteristically small convergence basins with any of the applied stabilising transformations.

in a typical increase of the fraction of converged seeds with the increasing value of parameter β in Eq. (1.3). For example, when detecting period-10 orbits of CHM using period-12 orbits as seeds, the fraction of seeds that converge to periodic orbits grows from 25-30% for small β (essentially the Newton-Raphson method) to about 70% for the optimal value of β.

5. Discussion and Conclusions. We have presented a new scheme for constructing stabilising transformations which can be used to locate periodic orbits in chaotic maps with the iterative scheme given by Eq. (1.3). The scheme is based on the understanding of the relationship between the stabilising transformations and the properties of eigenvalues and eigenvectors of the stability matrices of the periodic orbits. Of particular significance is the observation that only the unstable eigenvalues are important for determining the stabilising transformations. Therefore, unlike the original set of transformations proposed by Schmelcher and Diakonos, which grows with the system dimensionality as $2^n n!$, our set has the size of at most 2^k, where k is the maximum number of unstable eigenvalues (i.e. the maximum dimension of the unstable manifold). It is also apparent that, while the SD set contains a large fraction of transformations that do not stabilise any UPOs of a given system, all of our transformations stabilise a significant subset of UPOs. The dependence of the number of transformations on the dimensionality of the unstable manifold rather than on the system dimensionality is especially important in cases when we study low-dimensional chaotic dynamics embedded in a high-dimensional phase space. This is often the case in systems obtained from time-space discretisation of nonlinear partial differential equations (e.g. the Kuramoto-Sivashinsky equation). Application of the stabilising transformations approach to such high-dimensional chaotic systems will be the subject of our future work.

The new transformations were tested on two systems: a kicked double rotor map and three symmetrically coupled Hénon maps. We aimed to achieve a plausibly complete detection of periodic orbits of low periods up to as high a period as was computationally feasible. In both cases our algorithm was able to detect large numbers of UPOs with high degree of certainty that the sets of UPOs for each period were complete. We have used the symmetry of the systems in order to test the completeness of the detected sets. On the other hand, when the aim is to detect as many UPOs as possible without verifying the completeness, the symmetry of the system could be used to increase the efficiency of the detection of UPOs: once an orbit is detected, additional orbits can be located by applying the symmetry transformations.

One apparent drawback of the new scheme is that a small set of UPOs needs to be available for the construction of the stabilising transformation at the start of the detection process. With the systems studied so far, we had no problem detecting UPOs of low period using the standard Newton-Raphson method by setting $\beta = 0$ in Eq. (1.3). However, in systems where it is hard to detect even a single periodic orbit, it would be useful to be able to determine stabilising transformations without the knowledge of UPOs. Since the stabilising transformations depend mostly on the properties of the unstable subspace, and since the decomposition into stable and unstable subspaces can be defined at any, not just periodic, point on the chaotic set, it should be possible to estimate such properties and construct the stabilising transformations without the knowledge of the UPOs. The decomposition could be done, for example, in a process similar to that used in the subspace iteration algorithm [17], and a set of stabilising transformations, for example $\mathcal{C}_{\mathrm{SD}}$, could then be applied only within the unstable subspace. The feasibility of such a construction will be the topic

for future investigation.

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